

Mathematical Practices and the Role of Interactive Dynamic Technology

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Abstract:

The Common Core State Standards in Mathematics adopted by most of the states in the United States offer a set of mathematical practice standards as part of the expectations for all students. The practice standards suggest mathematical "habits of mind" teachers should cultivate in their students about ways of thinking and doing mathematics. With the teacher as a facilitator and using the right questions, dynamic interactive technology can be an effective tool in providing opportunities for students to engage in tasks that make these practices central in reasoning and doing the mathematics.

Introduction

Until 2010, each state in the United States had its own version of what mathematics was important to teach in K-12 classrooms and its own ways of measuring whether students were meeting their standards. The consequences of this were vast: enormous text books designed to meet every standard from any state but providing teachers little direction for making the text applicable in their state; assessments varying from multiple choice questions focused on procedures to using college entrance examinations as high school exit examinations. This variability in standards and assessments resulted in very different benchmarks for quality performance - one state's "A" was another state's "D". Comparing student performance on state assessments to the results of the National Assessment of Educational Progress, given to randomly selected students across the United States, makes this clear. Overall, individual states' level of satisfactory performance for students has been consistently lower than that of NAEP but varies greatly from state to state; for example in 2009, the difference in the percent of eighth grade (age 13-14) students who achieved a basic level of proficiency on the two tests was 11% in Massachusetts (43% to 32%) and 45% in New York (71% to 26%).

Deeming these results unacceptable and responding to the poor showing of United States students on TIMSS and the Programme for International Student Assessment (PISA), the Council of Chief State School Officers, the directors of education in each of the states, commissioned a set of national standards, the Common Core State Standards in Mathematics (CCSS), written by a small team of mathematicians and educators and released in 2010. As of April 2011, the standards were accepted by 42 of the 50 states. In addition to a set of content standards, one central and potentially significant component of the CCSS is a set of standards that focus on the mathematical "habits of mind" teachers should cultivate in their students about ways of thinking and doing mathematics, beginning in primary grades and continuing throughout secondary school. If implemented, these standards have the potential to significantly change how teachers in the United States approach teaching mathematics and what students take from their mathematics education. The eight practices are:

MP1: Make sense of problems and persevere in solving them

MP2: Reason abstractly and quantitatively

MP3: Construct viable arguments and critique the reasoning of others

MP4: Model with mathematics

- MP5: Use appropriate tools strategically
- MP6: Attend to precision
- MP7: Look for and make use of structure
- MP8: Look for and express regularity in repeated reasoning

Elaborations of the practice standards can be found at www.corestandards.org/assets/CCSSI_Math%20Standards.pdf. While the practices are part of the standards in the United States, in essence they seem universal. Cuoco and colleagues (1996) claim that the central aim of mathematics instruction should be to give students mental habits that allow them to develop a repertoire of general ways of thinking, strategies and tools they will need to use and understand in order to do mathematics and approaches that can be applied in many different situations. The mathematical practices are an attempt to make this visible to teachers – to call attention to ways of reasoning and sense making in mathematics.

The focus of this paper is to suggest how the practices can be used to leverage deeper understanding of core mathematical content by elaborating on two examples, one from calculus and one from mathematics. The discussion below describes some of the research about learning and how the use of dynamic interactive technology can support the implementation of the examples in classrooms.

Research and Dynamic Interactive Technology

Research suggests learning takes place when students engage in concrete experiences, observe reflectively, develop abstract conceptualizations based upon the reflection, and actively experiment or test the abstraction (Zull, 2002). The kind of learning that supports the transfer of concepts and ways of thinking occurs when students are actively involved in choosing and evaluating strategies, considering assumptions, and receiving feedback. They should encounter contrasting cases, noticing new features and identifying important ones (NRC, 1999). Dynamic interactive technology provides an environment in which these kinds of learning opportunities can take place. They allow students to deliberately take a mathematically meaningful action on a mathematical object, observe the consequences of that action, and reflect on the mathematical implications of those consequences, an "Action/Consequence" principle (Dick & Burrill, 2008).

In such environments, the task and role of the teacher are central as interactive technology alone is not sufficient for students to learn. Research about effective use of interactive applets in learning statistical concepts suggests teachers should engage students in activities that help them confront their misconceptions and provide them with feedback (delMas, Garfield, & Chance, 1999). Students' work needs to be both structured and unstructured to maximize learning opportunities; even a well-designed simulation is unlikely to be an effective teaching tool unless students' interaction with it is carefully structured (Lane & Peres 2006). In addition, students need to discuss observations after an activity to focus on important observations, become aware of missed observations, and reflect on how important observations are connected (Chance et al, 2007).

The following suggests how the mathematical practices might be realized in a calculus task. (Note that the figures were produced on a TI Nspire; could also be reproduced in any applet.)

Example 1: The Urn

Students are given an interactive file displaying an urn that can be filled with water using a clicker. A side panel displays a graph of the height vs. the volume as the urn is filled (Figure 1). To help students make sense of the problem and become familiar with the context (MP1: make sense of problems), they are asked questions such as, "Will the graph ever be a straight line? Why or why not?"; "If the urn were in the shape of a cone, what do you think the graph would look like? Why?"; "How would the graph of the height vs. the volume for a cone differ from the graph for an inverted cone?" Where is

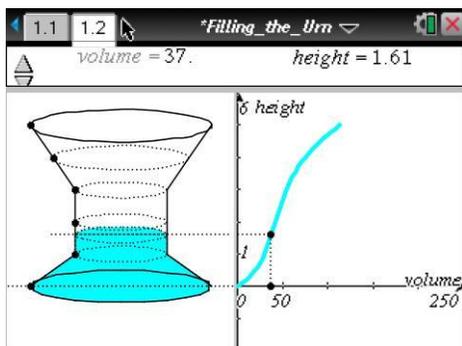


Figure 1 Filling the urn

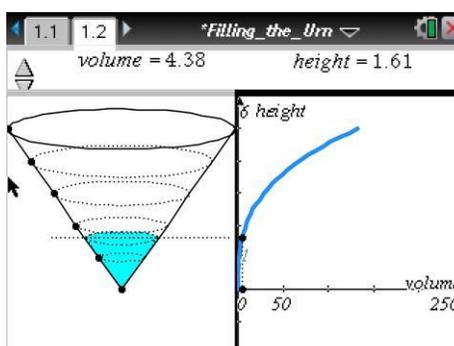


Figure 2 Filling a cone

the graph of height vs. volume the steepest? Explain your thinking." Once students have considered how the shape of the urn might affect the volume and made some conjectures (MP2: Reasoning), they "fill" the urn with water and check their reasoning. Students can be given time to experiment with different shapes and by describing them to their peers, can strengthen their understanding and ability to talk about the critical points of a function and the characteristics of its first and second derivatives.

Evidence from an analysis of student assessments in calculus indicated that many students struggle with several core calculus concepts. One of these seems to be the relationship among the characteristics of a function, its derivative and its second derivative, in particular with respect to multiple roots. The problem below, answered correctly by 28% of the students including 55% of those who earned top marks, is typical (College Board, 1998).

- If $f''(x) = x(x+1)(x-2)$, then the graph of f has inflection points when $x =$
A) -1 only B) 2 only C) -1 and 0 only D) -1 and 2 only E) -1, 0, and 2 only

To develop and support understanding, questions such as the following can be posed focus students on "contrasting cases, noticing new features and identifying important ones" (NRC, 1999): "What shape will produce a graph that has two points of inflection? Is it possible to have more than one shape that will produce such a graph?"; "What shape, if any, will produce a graph that is concave up with one point of inflection?"; "Predict the shape of the graph given an urn that is shaped like an hourglass, describe its characteristics, and defend your reasoning."

To provide students with feedback from both teachers and peers, a central feature of formative assessment (Black et al, 2004), classroom discussions should engage students in defending their solutions and sharing their graphs, preferably using a display that allows screen captures of student handhelds or computer screens (for example, the TI

Navigator). Students can be called on to look for similarities and differences in solutions, explain how another student might have been thinking about the problem, or decide which graphs are appropriate for a given question and why. The discussion can provide opportunities for students to engage in a variety of mathematical practices including:

- MP1: using visual representations to solve a problem; explain correspondences between tables and graphs; comparing approaches
- MP2: Stop and think about what the symbols represent in context
- MP3: Make conjectures; distinguish correct reasoning from that which is flawed
- MP4: interpret mathematical results in the context of the situation and reflect on whether the results make sense; identify important quantities in a practical situation
- MP5: use technology to visualize the results of varying assumptions, explore consequences, and compare predictions with data; use technological tools to explore and deepen understanding of concepts
- MP6: Communicate precisely to others.

Example 2: An Optimization Problem

An analysis of high stakes exit or end of course tests at the high school level (Dick & Burrill, 2009) suggests that students struggle with reasoning about graphs and about the mathematics. Problematic areas for students as opposed to experts include making connections among representations (Pierce, 2004; Stacey, 2005; Ramirez et al, 2005) and recognizing when to use certain techniques (NRC, 1999). The following task involves some of these elements: A sailboat has two masts, 5m and 12m. They are 24m apart. They must be secured to the same location using one continuous length of rigging. What is the least amount of rigging that can be used (figure 3)?

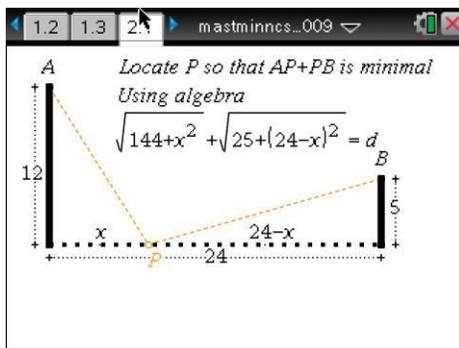


Figure 3 The Mast

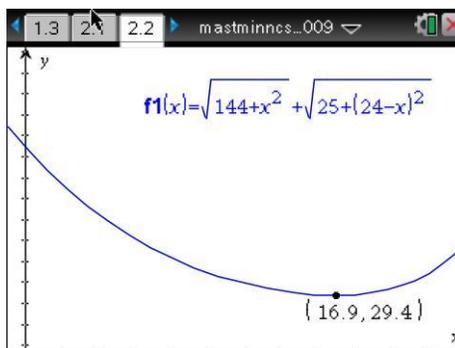


Figure 4 Using the Pythagorean Theorem

Students investigate the problem (MP1: making sense of the mathematics) by dragging point P making conjectures about possible locations for P that will minimize the amount of rigging. While different mathematical approaches to a solution are possible, fairly consistently in our work, students approach the solution algebraically using the Pythagorean theorem. Also fairly consistently, many of them graph the equation and trace to find a minimum point, which they suggest, without additional support for the claim, identifies the location of point P (figure 4). In less than half of the situations do students or teachers in workshops refer to calculus as a strategy to find the solution.

MP2 suggests mathematically proficient students should reason abstractly and quantitatively:

- make conjectures and build a logical progression of ideas
- determine domains to which an argument applies
- analyze situations by breaking them into cases
- recognize and use counterexamples
- compare effectiveness of two plausible arguments
- distinguish correct reasoning from that which is flawed and explain any flaws.

As students work through the strategies to find a solution, with the teacher as facilitator, they have the opportunity to engage in all of these mathematical practices. Graphing the equation results in a plausible suggestion for the location of P that will be the minimum but without further justification, it remains a conjecture. Students with a background in calculus should recognize the situation as one in which analyzing the derivative will yield critical points, which in conjunction with the second derivative, will lead to the solution. But it is also possible for students without a calculus background to reason from the context and implied domain for the function.

To reason about the situation means students have to recognize the difference between the x -coordinate of point P and the *distance* of point P from the first mast to the second (MP6: use clear definitions in discussion and in reasoning; state the meaning of symbols used), defining the *total length* to arrive at the same conclusion. The function defined by $L = \sqrt{x^2 + 25} + \sqrt{(24-x)^2 + 144}$ is constrained by the domain: $0 \leq x \leq 24$, which suggests that the length of the rigging is between the two extremes determined by the length of the deck and the length of a mast; the length is $\leq 12 + 24 = 36$ and the length is $\geq 5 + 24 = 29$. Reasoning about the sum of the parts of the graphs of the two hyperbolas over the domain can lead to the conclusion that the sum from $5 \leq x \leq 24$ can have only one minimum, and thus, the one illustrated on the graph is that point.

The task can be done geometrically by reflecting one of the masts over the horizontal deck line and connecting the top of one mast to the bottom of the reflected one (figure 5). Point P lies at the intersection of this line and the deck, reasoning that the shortest distance between two points is a straight line and the two small triangles are congruent.

Configuring the problem on a grid, dynamic interactive geometry software led one student to examine the point of intersection of the two diagonals (figure 6). She

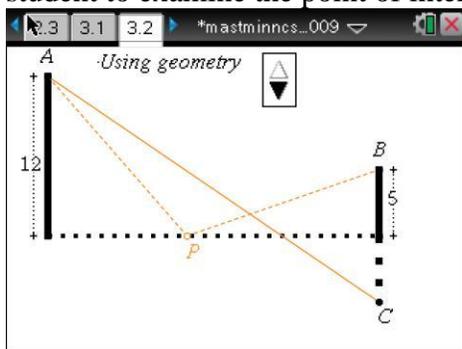


Figure 5 A transformation approach

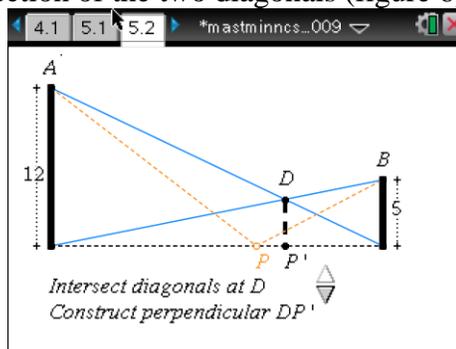


Figure 6 Diagonals

hypothesized that the x -value of the point of intersection might be that of point P, which has a nice proof that can be generalized (MP1: check answers by a different method; MP5: analyze graphs, functions and solutions generated by technology).

Conclusion

The two examples illustrate how mathematical "habits of mind" can emerge in a task, the way it is posed, and the way in which it is implemented, engaging students in thinking and reasoning about core mathematics. A task should be judged by the number of opportunities it affords to engage students in the mathematical practices. The role of the teacher is to frame the tasks and their implementation to maximize opportunities for learning. The questions teachers ask are central in bringing the practices – and ways of mathematical thinking and reasoning – to the foreground in the discussion. Technology is a tool that supports this work, providing students with opportunities to visualize the results of varying assumptions, explore the consequences, and compare predictions, explore and deepen understanding of concepts, and analyze graphs, functions and solutions generated by technology (MP5).

References

- Calculus Nspired. (2010). Math Nspired. Texas Instruments Education Technology. www.ti-mathnspired.com/login/?next=/
- Chance, B., Ben-Zvi, D., Garfield, J., & Medina, B. (2007). The role of technology in improving student learning in statistics. *Technology Innovations in Statistics*, 1(1).
- College Board (2000). The 1998 AP Calculus AB and AP Calculus BC Released Exams. New York NY: College Entrance Examination Board.
- Council of Chief State School Officers (CCSSO). (2010). Common Core State Standards. Council of Chief State School Officers and (National Governor's Association (NGA)
- Dick, T. & Burrill, G. (2009). Technology and teaching and learning mathematics at the secondary level: Implications for teacher preparation and development. Presentation Annual Meeting of Association of Mathematics Teacher Educators, Orlando FL.
- delMas, R., Garfield, J., & Chance, B. (1999) A model of classroom research in action: Developing simulation activities to improve students' statistical reasoning. *Journal of Statistics Education*, 7(3), www.amstat.org/publications/jse/secure/v7n3/delmas.cfm.
- Lane, D. M. & Peres, S. C. (2006) Interactive simulations in the teaching of statistics: Promise and pitfalls. *Proceedings of the Seventh Annual Meeting of the International Conference on the Teaching of Statistics*, Salvador, Brazil.
- National Assessment of Educational Progress. (2011). febp.newamerica.net/k12/MA
- National Research Council. (1999). *How People Learn: Brain, mind, experience, and school*. Bransford, J. D., Brown, A. L., & Cocking, R. R. (Eds.). Washington, DC: National Academy Press.
- Pierce, R. & Stacey, K. (2004) Learning to Use CAS: Voices from a Classroom. In M. Hoines & A. Berit Fuglestad (Eds.) *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education – PME 28*
- Ramirez, C., Oktac, A., Garcia, C. (2005). Las dificultades de los estudiantes en los sistemas de ecuaciones lineales con dos variables. Actas del XIII Congreso Nacional de Enseñanza de las Matemáticas, Acapulco, Guerrero.
- Stacey, K. (2005). Accessing results from research on technology in mathematics education. *Australian Senior Math Journal*, 19(1), pp 8-15.
- James Zull, (2002). *The art of changing the brain: Enriching the practice of teaching by exploring the biology of learning*. Alexandria, VA: Association for Supervision and Curriculum Development.